Eddy Current Computation by the FEM-SDBCI Method

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The hybrid FEM-SDBCI method is developed for the finite element computation of time-harmonic eddy current problems in open boundary domains. The method is similar to the well-known FEM-BEM, but it assumes a Dirichlet boundary condition on the truncation boundary instead of a Neumann one. Shorter solving times are obtained with respect to FEM-BEM.

Index Terms-- **Finite element method, boundary element method, integral equations, eddy currents.**

I. Introduction

Both FEM-BEM (Finite Element Method - Boundary Element Method) and FEM-DBCI (Dirichlet Boundary Condition Iteration) [1,2] couple a differential equation, which governs the interior problem, with an integral one which makes use of the free-space Green function and expresses the unknown boundary condition on the fictitious truncation boundary. The differences between the two methods are the following: in FEM-BEM an unknown Neumann condition is assumed on the truncation boundary, which coincides with the integration surface of the integral equation, whereas in FEM-DBCI an unknown Dirichlet boundary condition is assumed on the truncation boundary and another surface is used in the integral equation, so that singularities are avoided. Both the resulting global algebraic systems are partly sparse and partly dense.

In order to alleviate the major drawback of FEM-DBCI, that is, the insertion of some element layers between the integration and truncation surfaces, this paper presents a modified version of the method, named FEM-SDBCI (Singular DBCI), in which the two surfaces are coincident and the integral equation becomes singular.

II. The FEM-SDBCI Method

Consider an eddy current problem, in which a set of massive conductors are placed near a set of coils through which given time-harmonic source currents flow. Under the assumption of quasi-static time-harmonic steady-state behavior, the problem can be conveniently analyzed in terms of the electric field E by solving the differential equation:

$$\nabla\times\left(\nu\,\nabla\times\overline{E}\right)+j\omega\sigma\overline{E}=-j\omega\overline{J}_S \tag{1}$$

where ν is the magnetic reluctivity, σ the electric conductivity, ω the angular frequency and $\overline{J}_S$ the given current density in the coil regions. In order to apply FEM, the unbounded medium is truncated by means of a fictitious boundary Γ_F, enclosing all the eddy current conductors. Optionally some coils may be left outside. On Γ_F a non-homogeneous Dirichlet boundary condition is assumed:

$$\hat{n}\times\overline{E}=\overline{E}_F \tag{2}$$

where $\hat{n}$ is the outward versor normal to Γ_F and $\overline{E}_F$ is the component of the electric field along Γ_F. Discretizing the domain by means of tetrahedral edge elements, and applying the Galerkin method, the following matrix equation is derived:

$$\mathbf{ME}=\mathbf{N}_0-\mathbf{M}_F\mathbf{E}_F \tag{3}$$

where $\mathbf{M}$ and $\mathbf{M}_F$ are sparse matrices, $\mathbf{E}$ and $\mathbf{E}_F$ are the arrays of the field expansion coefficients for the internal and boundary edges, respectively, $\mathbf{N}_0$ is due to the internal source currents.

Another equation relating $\mathbf{E}$ to $\mathbf{E}_F$ is obtained by expressing the field on a point P on Γ_F by means of the integral:

$$\frac{\alpha}{4\pi}\overline{E}(P)=\overline{E}_{ext}(P)+\frac{1}{4\pi}\oint\!\!\!\oint_{\Gamma_F}\left(\frac{1}{r}\hat{n}'\times\nabla\times\overline{E}(P')+[\hat{n}'\times\overline{E}(P')]\times\nabla\frac{1}{r}+\hat{n}'\cdot\overline{E}(P')\nabla\frac{1}{r}\right)dS' \tag{4}$$

where r is the distance between points P and P', $\overline{E}_{ext}$ is the field due to the coil source currents external to Γ_F, and α is the solid angle of the domain at P. The expansion coefficient E_m relative to an edge e_m lying on the fictitious boundary is expressed as:

$$\frac{\alpha}{4\pi}E_m=\frac{1}{L_m}\int_{e_m}\overline{E}_{ext}(P_F)\cdot\hat{t}_m\,ds+\frac{1}{4\pi L_m}\cdot\sum_k\iint_{T_k}\int_{e_m}[\frac{1}{r}\hat{n}\times\nabla\times\overline{E}+(\hat{n}\times\overline{E})\times\nabla\frac{1}{r}+\hat{n}\cdot\overline{E}\nabla\frac{1}{r}]\cdot\hat{t}_m\,ds\,dS \tag{5}$$

where T_k is the k-th triangular patch on the fictitious boundary, coming from the tetrahedral mesh of the domain. Both the double integral on the triangle T_k and the line integral on the edge e_m are computed by means of the Gauss quadrature. The integration accuracy can be selected according to the following rule. Let L_k be the length of the longest edge of the triangle T_k on Γ_F, L_m the length of the edge on the fictitious boundary, $L=\max(L_{max},L_m)$ and d the distance between their centers; then for $L/d\leq0.2$ a one-point quadrature is used on both the triangle and the edge; for $0.2<L/d\leq1.1$ three Gauss points are used on the triangle and two points on the edge; otherwise six points are used on the

triangle and three on the edge. This rule has proved to be a good tradeoff between accuracy and speed, as extensive numerical investigations have shown.

The singularities arising in the integrand function in (5) are overcome by means of analytical formulas [4]. Finally we get:

$$\mathbf{HE}_{\mathrm{F}} = \mathbf{E}_{\mathrm{ext}} + \mathbf{GE} \quad (6)$$

where **H** and **G** are dense matrices.

Combining (3) and (6), the global linear algebraic system of the FEM-SDBCI method is formed:

$$\begin{bmatrix} \mathbf{M} & \mathbf{M}_{\mathrm{F}} \\ -\mathbf{G} & \mathbf{H} \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ \mathbf{E}_{\mathrm{F}} \end{bmatrix} = \begin{bmatrix} \mathbf{N}_0 \\ \mathbf{E}_{\mathrm{ext}} \end{bmatrix} \quad (7)$$

In order to solve (7) we consider the reduced system:

$$\mathbf{A}\mathbf{E}_{\mathrm{F}} = \mathbf{B} \quad (8)$$

where:

$$\mathbf{A} = \mathbf{H} + \mathbf{G}\mathbf{M}^{-1}\mathbf{M}_{\mathrm{F}} \quad (9)$$

$$\mathbf{B} = \mathbf{E}_{\mathrm{ext}} + \mathbf{G}\mathbf{M}^{-1}\mathbf{N}_0 \quad (10)$$

Matrix **A** and vector **B** in (8) are not directly available. However, the vector **B** is simply built as follows: 1) assume a zero initial guess $\mathbf{E}_F=\mathbf{0}$; 2) solve (3) for **E** by means of the conjugate gradient (CG) solver to obtain $\mathbf{E}=\mathbf{M}^{-1}\mathbf{N}_0$; 3) compute $\mathbf{B}=\mathbf{E}_{ext}+\mathbf{GE}$. Matrix **A** can be used to perform matrix-vector multiplication $\mathbf{AE}_F$, as follows: 1) given a vector $\mathbf{E}_F$; 2) solve (3) with $\mathbf{N}_0=\mathbf{0}$ to obtain $\mathbf{E}=-\mathbf{M}^{-1}\mathbf{M}_F\mathbf{E}_F$; 3) compute $\mathbf{AE}_F=\mathbf{HE}_F-\mathbf{GE}$.

Then several non-stationary iterative CG-like solvers for non symmetric matrices can be used to solve (8). GMRES should be preferred, since it performs a true minimization of the residual and hence of the number of steps. Note that the major drawbacks of GMRES, that is the computing time and memory required to compute and store the orthonormal basis, in this case are not very heavy, because GMRES works on the reduced system (8), in which the number of unknowns is the number of the edges on the fictitious boundary.

II. A Numerical Example

The system analysed is the classical Bath plate with two holes [5]. A conducting ladder ($\sigma = 32.78\cdot10^2$ S/m) with two holes (length l=110 mm, width w=60 mm, height h=6.35 mm, central column and yoke width 10 mm, lateral column 20 mm) is under (s=15 mm) a toroidal coil (1260 Amp turns, frequency f=50 Hz) having a square section of side 20 mm, internal radius 20 mm and axis equations x =w/2, y =l/2+s (Fig. 1).

FEM-SDBCI was applied leaving the coil outside the fictitious boundary, which coincides with the conductor surface. The x=w/2 plane is a symmetry one, so only half of the original domain needs to be meshed, by imposing a homogeneous Dirichlet boundary condition on such a plane. The mesh consists of 3360 tetrahedra and 5546 edges, 2786 of which lie on the conductor surface. The GMRES solver converges in 11 iteration steps with an end iteration tolerance of 0.1 per cent. Fig. 2 shows the eddy currents on the z=h/2 plane. A check of the Kirchhoff law was made by evaluating the three currents through the three sections with the symmetry plane: $I_1=I_2+I_3$; an acceptable fulfilment was obtained.

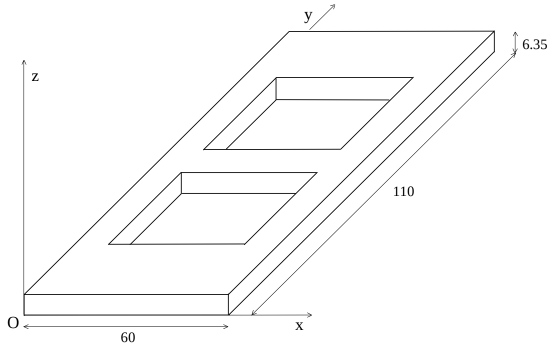

Fig. 1. The Bath plate analyzed.

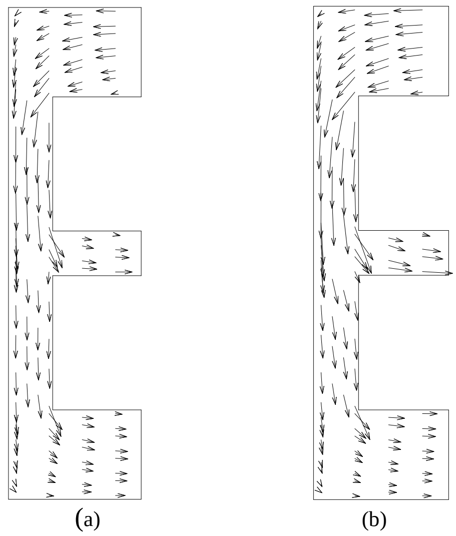

Fig. 2. Real (a) and imaginary (b) parts of the eddy currents in the Bath plate.

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